

RECENT PROGRESS IN SPIN CALCULATIONS IN THE POST-NEWTONIAN FRAMEWORK AND APPLICATIONS

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Recently we derived the next-to-next-to-leading order post-Newtonian Hamiltonians at spin-orbit and spin(1)-spin(2) level for a binary system of compact objects. In this talk the derivation of them will be shortly outlined at an introductory level. We will also discuss some checks of our (complicated and long) results in the first part of the talk. In the second part we will show how to apply our results to the calculation of the last stable circular orbit of such a binary system of black holes or neutron stars.

Keywords: Post-Newtonian approximation; Spinning binaries; Equations of motion

Astrophysical binary systems usually contain spinning components. With the dawn of gravitational wave astronomy approaching, spin contributions to the equations of motion become very important. Recent years have seen a lot of progress in this direction, see Ref. 1 for a summary of literature in this field. The present work uses an extension of the ADM formalism² to spinning objects linear in the spins.^{3,4} The Hamiltonians in question are the next-to-next-to-leading order (NNLO) spin-orbit (SO) and NNLO spin(1)-spin(2) (SS) ones.^{5,6} Details on the corresponding formal 3PN calculations will be covered in a recent manuscript.^{1,7}

For brevity we provide the Hamiltonians in the center-of-mass frame. The full results are given in the original publications.^{1,5-7} In this frame and in dimensionless quantities^{8,9} they are given by

$$\begin{aligned}
 H_{\text{COM SO}}^{\text{NNLO}} = & \frac{1}{4r_{12}^5} \left[21\sqrt{1-4\eta}(\eta+1)(\mathbf{L} \cdot \boldsymbol{\Delta}) + \frac{1}{2}(-2\eta^2 + 33\eta + 42)(\mathbf{L} \cdot \boldsymbol{\Sigma}) \right] \\
 & + \frac{\eta}{32r_{12}^4} \left[-\sqrt{1-4\eta}((256+45\eta)(\mathbf{n}_{12} \cdot \hat{\mathbf{p}})^2 + (314+39\eta)\hat{\mathbf{p}}^2)(\mathbf{L} \cdot \boldsymbol{\Delta}) \right. \\
 & \quad \left. + ((-256+275\eta)(\mathbf{n}_{12} \cdot \hat{\mathbf{p}})^2 + (-206+73\eta)\hat{\mathbf{p}}^2)(\mathbf{L} \cdot \boldsymbol{\Sigma}) \right] \\
 & + \frac{\eta}{32r_{12}^3} \left[\sqrt{1-4\eta}(15(\mathbf{n}_{12} \cdot \hat{\mathbf{p}})^4 + 3(9\eta-4)(\mathbf{n}_{12} \cdot \hat{\mathbf{p}})^2\hat{\mathbf{p}}^2 \right. \\
 & \quad \left. + 2(22\eta-9)(\hat{\mathbf{p}}^2)^2)(\mathbf{L} \cdot \boldsymbol{\Delta}) - (15(2\eta-1)(\mathbf{n}_{12} \cdot \hat{\mathbf{p}})^4 \right. \\
 & \quad \left. + 3(6\eta^2-11\eta+4)(\mathbf{n}_{12} \cdot \hat{\mathbf{p}})^2\hat{\mathbf{p}}^2 + 2(5\eta^2-3\eta+2)(\hat{\mathbf{p}}^2)^2)(\mathbf{L} \cdot \boldsymbol{\Sigma}) \right], \quad (1)
 \end{aligned}$$

$$\begin{aligned}
H_{\text{COM SS}}^{\text{NNLO}} = & \eta \left\{ \frac{1}{4r_{12}^5} \left[(79\eta + 105)(\mathbf{n}_{12} \hat{\mathbf{S}}_1)(\mathbf{n}_{12} \hat{\mathbf{S}}_2) - (63 + 19\eta)(\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_2) \right] \right. \\
& + \frac{1}{r_{12}^4} \left[- \left(\frac{303}{4}\eta(\mathbf{n}_{12} \hat{\mathbf{p}})^2 + \left(\frac{125}{4}\eta + 9 \right) \hat{\mathbf{p}}^2 \right) (\mathbf{n}_{12} \hat{\mathbf{S}}_1)(\mathbf{n}_{12} \hat{\mathbf{S}}_2) \right. \\
& \quad \left(- \left(18 + \frac{25}{4}\eta \right) (\mathbf{n}_{12} \hat{\mathbf{p}})^2 + \left(9 + \frac{47}{2}\eta \right) \hat{\mathbf{p}}^2 \right) (\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_2) \\
& \quad - \frac{9}{4}(7\eta + 4)(\hat{\mathbf{p}} \hat{\mathbf{S}}_1)(\hat{\mathbf{p}} \hat{\mathbf{S}}_2) \\
& \quad + \left(34\eta + \frac{27}{2} \right) (\mathbf{n}_{12} \hat{\mathbf{p}})((\hat{\mathbf{p}} \hat{\mathbf{S}}_1)(\mathbf{n}_{12} \hat{\mathbf{S}}_2) + (\mathbf{n}_{12} \hat{\mathbf{S}}_1)(\hat{\mathbf{p}} \hat{\mathbf{S}}_2)) \\
& \quad \left. + \frac{3}{2}\sqrt{1-4\eta}(\eta+3)(\mathbf{n}_{12} \hat{\mathbf{p}})((\hat{\mathbf{p}} \hat{\mathbf{S}}_1)(\mathbf{n}_{12} \hat{\mathbf{S}}_2) - (\mathbf{n}_{12} \hat{\mathbf{S}}_1)(\hat{\mathbf{p}} \hat{\mathbf{S}}_2)) \right] \\
& + \frac{1}{r_{12}^3} \left[\frac{1}{8} \left(105\eta^2(\mathbf{n}_{12} \hat{\mathbf{p}})^4 + 15\eta(3\eta-2)(\mathbf{n}_{12} \hat{\mathbf{p}})^2 \hat{\mathbf{p}}^2 \right. \right. \\
& \quad \left. + \frac{3}{2}(10\eta^2 + 13\eta - 6)(\hat{\mathbf{p}}^2)^2 \right) (\mathbf{n}_{12} \hat{\mathbf{S}}_1)(\mathbf{n}_{12} \hat{\mathbf{S}}_2) \\
& \quad + \frac{1}{8} \left(-3(8\eta^2 - 37\eta + 12)(\mathbf{n}_{12} \hat{\mathbf{p}})^2 \hat{\mathbf{p}}^2 + (7\eta^2 - 23\eta + 9)(\hat{\mathbf{p}}^2)^2 \right) (\hat{\mathbf{S}}_1 \hat{\mathbf{S}}_2) \\
& \quad + \frac{1}{4} \left(9\eta^2(\mathbf{n}_{12} \hat{\mathbf{p}})^2 + \frac{1}{2}(4\eta^2 + 25\eta - 9)\hat{\mathbf{p}}^2 \right) (\hat{\mathbf{p}} \hat{\mathbf{S}}_1)(\hat{\mathbf{p}} \hat{\mathbf{S}}_2) \\
& \quad - \frac{3}{8} \left(+15\eta^2(\mathbf{n}_{12} \hat{\mathbf{p}})^2 + \frac{1}{2}(10\eta^2 + 21\eta - 9)\hat{\mathbf{p}}^2 \right) (\mathbf{n}_{12} \hat{\mathbf{p}}) \\
& \quad \quad \times ((\hat{\mathbf{p}} \hat{\mathbf{S}}_1)(\mathbf{n}_{12} \hat{\mathbf{S}}_2) + (\mathbf{n}_{12} \hat{\mathbf{S}}_1)(\hat{\mathbf{p}} \hat{\mathbf{S}}_2)) \\
& \quad \left. + \frac{9}{16}\sqrt{1-4\eta}(1-2\eta)(\mathbf{n}_{12} \hat{\mathbf{p}})((\hat{\mathbf{p}} \hat{\mathbf{S}}_1)(\mathbf{n}_{12} \hat{\mathbf{S}}_2) - (\mathbf{n}_{12} \hat{\mathbf{S}}_1)(\hat{\mathbf{p}} \hat{\mathbf{S}}_2)) \right] \left. \right\}. \tag{2}
\end{aligned}$$

There $\mathbf{\Delta} = \hat{\mathbf{S}}_1 - \hat{\mathbf{S}}_2$ and $\mathbf{\Sigma} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$ the differences and sums of the spin vectors, \mathbf{L} is the orbital angular momentum $\mathbf{L} = r_{12}\mathbf{n}_{12} \times \hat{\mathbf{p}}$ and $\eta = m_1 m_2 / (m_1 + m_2)^2$ is the symmetric mass ratio. The spins $\hat{\mathbf{S}}_a$, relative position \mathbf{r}_{12} and linear momentum $\hat{\mathbf{p}}$ variables fulfill the usual canonical Poisson brackets, where $r_{12} = |\mathbf{r}_{12}|$ and $\mathbf{n}_{12} = \mathbf{r}_{12}/r_{12}$.

Certain tests of these results were performed. Most important, kinematical consistency checks by using the global (post-Newtonian approximate) Poincaré algebra are applied. This requires the appropriate center-of-mass vectors.^{1,5,6} Furthermore, we checked our Mathematica code by re-deriving the 3PN ADM point-mass Hamiltonian including the corresponding ultraviolet analysis in dimensional regularization. Also the Hamiltonian of a test-spin moving in a stationary Kerr spacetime is obtained by rather simple approach and used to check parts of the mentioned results.

As an application the last stable circular orbit was determined via an ansatz

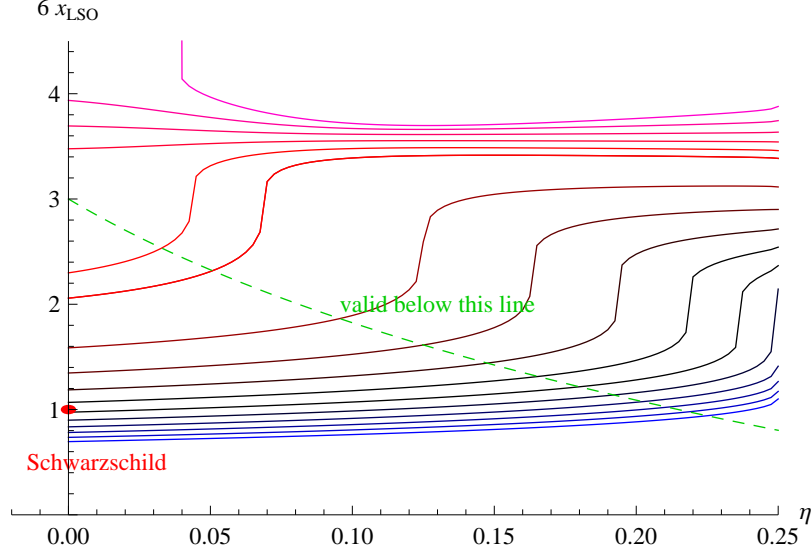


Fig. 1. Last stable circular orbit for $S = 0.1$ plotted for different symmetric mass ratios η and Kerr spins a . (From lower to upper lines $a = -1.0$ to $a = 0.84$. The two uppermost plots contain the cases $a = 0.8$ and $a = 0.84$ respectively. The difference in Kerr spin between all other plots is $\Delta a = 0.2$.) The ellipse on the vertical axis denotes the last stable circular orbit of a testmass orbiting a Schwarzschild black hole.

modified by η -dependent coefficients, which were matched to the post-Newtonian expansion derived from the mentioned Hamiltonians,⁹ see Fig. 1. Notice that the approximation brakes down above the dashed line.

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